# CHARACTERISTICALLY NILPOTENT LIE ALGEBRAS

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**Resumen**

We provide an overview of the theory of characteristically nilpotent Lie algebras and the methods used for their determination. We also prove the density of these algebras in the irreducible components of the variety $\mathfrak{N}^8$ and construct nonfiliform characteristically nilpotent Lie algebras. The concept of characteristic nilpotence is generalized to the sequence of algebras of derivations.

## 1 Introduction

The starting point for the theory of characteristically nilpotent Lie algebras was probably the article by Lister and Dixmier [15] in 1957, where they gave the first example of a nilpotent Lie algebra all whose derivations are nilpotent, showing that the converse of Jacobson's theorem [24], which establishes that each Lie algebra on a field of characteristic zero having a nondegenerate derivation is nilpotent, is false. In the 60' many authors tried to determine the structure of the algebras of derivations of nilpotent Lie algebras. However, this problem is of extreme difficulty because the candidates to derivation algebras are almost all algebras, from nonsolvable to nilpotent Lie algebras. Probably the best results in this way correspond to the derivations of metabelian Lie algebras, studied by Leger and Luks [30].
In 1970 Dyer [17] gave an example of a nine dimensional characteristically nilpotent Lie algebra with nilpotent automorphism group, and the

first example of such an algebra in dimension 7 was obtained in 1972 by Favre [18]. This was in fact the lowest dimensional characteristically nilpotent Lie algebra known until then. With only isolated examples these algebras seemed to be scarce, however characteristically nilpotent Lie algebras form an important class within the nilpotent Lie algebras, and their study and determination is of importance for the classification theory and the study of the variety of laws $\mathfrak{L}^n$ and the variety of nilpotent Lie algebra laws $\mathfrak{N}^n$.

Luks [32] and Yamaguchi [41] have shown that characteristically nilpotent Lie algebras exist for any dimension $\geq 7$. The first used numerical methods to construct such algebras, while the second constructed examples that are valid for any odd dimension. Though there are a lot of papers dedicated to the determination of characteristically nilpotent Lie algebras, the most results correspond to filiform Lie algebras [25], i.e, nilpotent algebras with maximal nilpotence index. The cohomology theories play an essential role in this treatment, because the use of certain deformations allows to prove that almost all Lie algebras derived from the nilradical of Borel subalgebras of complex Lie algebras are characteristically nilpotent [27]. The cohomology is also used to construct large familes of filiform characteristically nilpotent Lie algebras [28].

In studying the varieties of laws, the characteristically nilpotent algebras have shown their importance in the determination of irreducible components. For example, in dimension 7 there are two components, the first formed by filiform Lie algebras and the second generated by the orbit closure of a family of characteristically nilpotent Lie algebras [5].

The main problem in the study of characteristically nilpotent Lie algebras is the determination of conditions for an algebra of derivations to be nilpotent: for an arbitrary nilpotent Lie algebra the structure of the algebra of derivations can variate from representations of the special linear algebras $\mathfrak{sl}_n$ to certain nilpotent Lie algebras.

## 1.1 The variety $\mathfrak{N}^n$ of nilpotent Lie algebra laws

Let $\mathfrak{L}^n$ ( respectively $\mathfrak{N}^n$) be the set of Lie algebras ( nilpotent Lie algebras) of dimension $n$. Fixing a basis $\{e_i\}$ of $\mathbb{C}^n$ we can identify the law with its structure constants $C_{ij}^{k}$ respect to this basis. The Jacobi identity

$$\sum_{l=1}^{n} C_{ij}^{l}C_{kl}^{s} + C_{jk}^{l}C_{il}^{s} + C_{ki}^{l}C_{jl}^{s} = 0, \;\; 1 \leq s \leq n, \; 1 \leq i \leq j < k \leq n$$

and the antisymmetry give an algebraic variety of $\mathbb{C}^{\frac{n^3-n^2}{2}}$. It follows immediately that the nilpotent laws $\mathfrak{N}^n$ define a closed subset in $\mathfrak{L}^n$. The topology on $\mathfrak{L}^n$ is either the Zariski topology or the induced topology from $\mathbb{C}^{\frac{n^3-n^2}{2}}$. Let $\mathcal{O}(\mu)$ be the orbit of a law by the the action of the linear group $GL(n,\mathbb{C})$ given by the changes of basis.

**Definition 1** *If the orbit $\mathcal{O}(\mu)$ is open in $\mathfrak{L}^n$ (resp. $\mathfrak{N}^n$) the law $\mu$ is rigid in $\mathfrak{L}^n$ ( resp. $\mathfrak{N}^n$).*

Now, if the law $\mu$ is rigid, the closure of its orbit[1] defines an irreducible component of $\mathfrak{L}^n$ ( or $\mathfrak{N}^n$, but in this case the law needs not to be rigid in $\mathfrak{L}^n$).

**Remark 2** *From now on, we are only interested in the variety of nilpotent Lie algebra laws $\mathfrak{N}^n$. For dimension $n \leq 6$ we know that the variety is irreducible [39]. For $n \geq 11$ Vergne [40] proved the existence of at least two irreducible components. In fact, the variety $\mathfrak{N}^n$ is reducible for any $n \geq 7$ [6]. We observe however that the determination of components based on rigid laws fails for the variety $\mathfrak{N}^n$, for there are no known nilpotent rigid laws in $\mathfrak{N}^n$ ( and $\mathfrak{L}^n$ ) for $n \geq 9$. The determination of components is then approached by considering parametrized families of nilpotent Lie algebras which give components and separating the laws by its characteristic sequence.*

Let $\mathfrak{g}_n = (\mathbb{C}^n, \mu)$ be a nilpotent Lie algebra. For each $X \in \mathfrak{g}_n - C^1\mathfrak{g}_n$ we denote $c(X)$ the ordered sequence of a similitude invariant of the nilpotent operator $ad_{\mathfrak{g}_n}(X)$; i.e, the ordered sequence of degrees of irreducible factors of the minimal polynomial for the adjoint operator. We order these sequences lexicographically.

[1]The closure in both topologies coincide, for we are working on $\mathbb{C}$.

**Definition 3** *The characteristic sequence of $\mathfrak{g}_n$ is an isomorphism invariant $c(\mathfrak{g}_n)$ defined by*

$$c(\mathfrak{g}_n) = \max_{X \in \mathfrak{g}_n - C^1\mathfrak{g}_n} \{c(X)\}$$

*where $C^1\mathfrak{g}_n$ is the derived algebra. A nonzero vector $X \in \mathfrak{g}_n - C^1\mathfrak{g}_n$ satisfying $c(X) = c(\mathfrak{g}_n)$ is called characteristic vector.*

The characteristic sequence is one of the most valuable invariants concerning nilpotent Lie algebras. It allows for example to classify the nilpotent algebras in dimension seven [3]. Here it will be useful in obtaining deductions about the irreducible components of the variety $\mathfrak{N}^n$. For this purpose we recall elementary definitions from the internal set theory ( I.S.T:, see [5],[22]).

**Definition 4** *A perturbation $\mu$ of a law $\mu_0 \in \mathfrak{L}^n$ ( $n$ standard) is a law such that*

$$\mu(e_i, e_j) \sim \mu_0(e_i, e_j), \ 1 \leq i \leq j \leq n$$

*for the standard fixed basis of $\mathbb{C}^n$.*

The classical equivalence of a perturbation is the deformation with small parameter [34]. In fact we will use this concept of deformation for the construction of characteristically nilpotent filiform Lie algebras. They are of great importance in relation with the characteristic sequence, for the invariants of nilpotent characterising the irreducible components are those which are stable by perturbations [5].

**Proposition 5** *For any standard law $\mu_0 \in \mathfrak{L}^n$ and each perturbation $\mu$ we have $c(\mu) \geq c(\mu_0)$*

As a consequence we obtain that the subsets

$$U_s^n = \{\mu \in \mathfrak{N}^n \mid c(\mu) \geq s\}$$

for a fixed characteristic sequence $(1, .., 1) \leq s \leq (n-1, 1)$ are open in the variety $\mathfrak{N}^n$. In particular, if the nilindex is maximal, we obtain the open set of filiform Lie algebras $\mathfrak{F}^n$. Then, if the open set of filiform is reducible, its components will give components of the variety $\mathfrak{N}^n$, though there are components given by nonfiliform laws [36].

## 1.2 Cohomology of nilpotent Lie algebras

Let $\mathfrak{g}$ be a nilpotent Lie algebra of nilindex $p$ and let

$$C^0\mathfrak{g} = \mathfrak{g}, \quad C^k\mathfrak{g} = \left[C^{k-1}\mathfrak{g}, \mathfrak{g}\right], \ 1 \leq k$$
$$C_0\mathfrak{g} = 0, \quad C_k\mathfrak{g} = \{x \in \mathfrak{g} \mid [x, \mathfrak{g}] \subset C_{k-1}\mathfrak{g}\}, \ 1 \leq k$$

be respectively the descending central and the ascending central sequences of $\mathfrak{g}$. We define

$$F_i\mathfrak{g} = \mathfrak{g} \text{ if } i \leq 1, \quad F_i\mathfrak{g} = C^{i-1}\mathfrak{g} \text{ if } i > 1,$$
$$T_i\mathfrak{g} = \mathfrak{g} \text{ if } i \leq 1, \quad T_i\mathfrak{g} = C_{p+1-i}\mathfrak{g} \text{ if } i > 1$$

Following relation is satisfied:

$$[F_i\mathfrak{g}, T_j\mathfrak{g}] \subset T_{i+j}\mathfrak{g}, \quad i, j \in \mathbb{Z}$$

The algebra $\mathfrak{g}$ is filtered through the sequence $\{F_i\mathfrak{g}\}$ and the adjoint module $\mathfrak{g}$ is filtered considering the sequence $\{T_i\mathfrak{g}\}$. These filtrations are called the usual filtrations for the Lie algebra $\mathfrak{g}$ and the adjoint module $\mathfrak{g}$. Thus these filtrations can be translated to the cohomology to obtain filtered spaces of cochains, cocycles, coboundaries and cohomology. The standard notations are

$$\left\{F_iC^j(\mathfrak{g}, \mathfrak{g})\right\}, \ \left\{F_iZ^j(\mathfrak{g}, \mathfrak{g})\right\}, \ \left\{F_iB^j(\mathfrak{g}, \mathfrak{g})\right\}, \ \left\{F_iH^j(\mathfrak{g}, \mathfrak{g})\right\}$$

Now suppose that the Lie algebra $\mathfrak{g}$ and the adjoint module $\mathfrak{g}$ are graded and that the filtrations associated with these graduations coincide with the usual filtrations. Then the cochains are also graded

$$C^j(\mathfrak{g}, \mathfrak{g}) = \bigoplus_{i \in \mathbb{Z}} C^j_i(\mathfrak{g}, \mathfrak{g})$$

from which we obtain graduations in the other mentioned spaces. We observe that the associated filtrations coincide with the usual ones for these spaces.

**Remark 6** *It is known that the space of 2-cocycles $Z^2(\mathfrak{g}, \mathfrak{g})$ is identified with the Zariski tangent space to $\mathfrak{g}$ on the variety of Lie algebra laws $\mathfrak{L}^n$ and the coboundaries $B^2(\mathfrak{g}, \mathfrak{g})$ to the tangent space on the orbit $\mathcal{O}(\mathfrak{g})$.*

*The filtrations allow to obtain the tangent variety to $\mathfrak{g}$ on the variety of nilpotent Lie algebra laws: it coincides with the vector space of cocycles whose cohomology class belong to $F_{p-q}(\mathfrak{g},\mathfrak{g})$, where $p$ is the nilindex and $q \geq p$. More precisely:*

**Proposition 7** *Let $q \geq p$ and $\mathfrak{g}_n$ be an $n$-dimensional nilpotent Lie algebra with associated law $\mu$ and nilindex $p$. Let $\psi_1 \in Z^2(\mu,\mu)$. Then there exists a coboundary $df \in B^2(\mu,\mu)$ such that $\psi_1 = df + \psi_2$ with $\psi_2$ a cocycle belonging to $F_{p-q}(\mu,\mu)$.*

**Corollary 8** *Let $\mathfrak{g} = (\mathbb{C}^n,\mu)$ be a nilpotent Lie algebra of nilindex $p$. Then the Zariski tangent space to $\mathfrak{g}$ in $\mathfrak{N}_p^n$ is the subspace $W$ whose elements are cocycles $\varphi \in Z^2(\mu,\mu)$ such that their cohomology class belongs to $F_{q-p}H^2(\mu,\mu)$.*

This theory is highly developed and provides important techniques for the study of nilpotent Lie algebras [16].

## 2 Characteristically nilpotent Lie algebras

We begin with the basic definitions and properties of characteristically nilpotent Lie algebras. Let $\mathfrak{g}_n$ be an $n$-dimensional complex Lie algebra and let $Der(\mathfrak{g}_n)$ be its Lie algebra of derivations. We define a generalization of the central descending sequence by the rule

$$\mathfrak{g}_n^{[1]} = Der(\mathfrak{g}_n)(\mathfrak{g}_n) = \{X \in \mathfrak{g}_n \mid X = f(Y),\ f \in Der(\mathfrak{g}_n),\ Y \in \mathfrak{g}_n\}$$

and

$$\mathfrak{g}_n^{[k]} = Der(\mathfrak{g}_n)\left(\mathfrak{g}_n^{[k-1]}\right),\ k > 1$$

This sequence was first considered by Dixmier and Lister [15].

**Definition 9** *A Lie algebra $\mathfrak{g}_n$ is called characteristically nilpotent if there exists an integer $m$ such that $\mathfrak{g}_n^{[m]} = 0$.*

We remark that for dimension $\leq 6$ there do not exist characteristically nilpotent Lie algebras. This fact is either deducible from the classifications given [33], but there are also structural obstructions for these dimensions that explain that 7 is the first dimension having such algebras. The following criterions can be found in [29].

**Lemma 10** *If $\mathfrak{g}_n$ is characteristically nilpotent, then*

1. *the center $Z(\mathfrak{g}_n)$ of $\mathfrak{g}_n$ is contained in the derived subalgebra $C^1\mathfrak{g}_n = [\mathfrak{g}_n, \mathfrak{g}_n]$*
2. *$C^2\mathfrak{g}_n \neq 0$.* ■

**Remark 11** *As a consequence of this lemma there do not exist metabelian characteristically nilpotent Lie algebras.*

**Example 12** *If* 1. *is not satisfied, then there is an element $x_0 \in Z(\mathfrak{g})$ such that $x_0 \notin C^1\mathfrak{g}$. Thus, if $f \in Der(\mathfrak{g})$ is a derivation of $\mathfrak{g}$, there is no condition on $f(x_0)$. Then we can take $f(x_0) = x_0$, from which follows that $Der(\mathfrak{g})$ is not nilpotent.*

The next proposition provides an equivalent definition of characteristic nilpotence which is usually taken.

**Proposition 13** *A complex Lie algebra $\mathfrak{g}_n$ $(n > 1)$ is characteristically nilpotent if and only if $Der(\mathfrak{g}_n)$ is nilpotent.*

**Proof.** The necessity follows immediately from the definition. Suppose that $Der(\mathfrak{g}_n)$ is nilpotent. We have the decomposition into weight spaces $\mathfrak{g}_n = (\mathfrak{g}_n)_0 \bigoplus \sum (\mathfrak{g}_n)_\alpha$, where

$$(\mathfrak{g}_n)_\alpha = \left\{X \in \mathfrak{g}_n \mid (f - \alpha.id)^k(X) = 0 \text{ for an integer } k \geq 1,\ f \in Der(\mathfrak{g}_n)\right\}$$

It is obvious that $\left[(\mathfrak{g}_n)_\alpha, (\mathfrak{g}_n)_\beta\right] \subset (\mathfrak{g}_n)_{\alpha+\beta}$ for any weights $\alpha, \beta$, where the last space is zero if the sum of weights is not a weight. As any $(\mathfrak{g}_n)_\alpha$ is an ideal, we have the relation

$$\left[(\mathfrak{g}_n)_\alpha, (\mathfrak{g}_n)_\beta\right] \subset (\mathfrak{g}_n)_\alpha \cap (\mathfrak{g}_n)_\beta \cap (\mathfrak{g}_n)_{\alpha+\beta}$$

We denote the unitary component[2] of the previous decomposition as $(\mathfrak{g}_n)_*$. The $\mathfrak{g}_n = (\mathfrak{g}_n)_0 \bigoplus (\mathfrak{g}_n)_*$, and the unitary component is a central ideal. As the Lie algebra $\mathfrak{g}_n$ is clearly nilpotent, it follows that either the

[2] As the given decomposition is the Fitting decomposition of $\mathfrak{g}_n$ respect to the algebra of derivations.

zero or the unitary component reduces to $\{0\}$. In fact, let $x \in (\mathfrak{g}_n)_*$ and $V$ a complement of $\mathbb{C}x$ in the unitary component. The zero component is nilpotent, so its center contains at least a nonzero element $z$. Let $f_1, f_2 \in Der(\mathfrak{g}_n)$ be the derivations defined by

$$f_1((\mathfrak{g}_n)_0) = 0, \quad f_1(x) = z, \quad f_1(V) = 0$$
$$f_2((\mathfrak{g}_n)_0) = 0, \quad f_2(z) = z, \quad f_2(V) = 0$$

It follows $[f_1, f_2] = f_1$, so that the adjoint operator $ad(f_1)$ in $Der(\mathfrak{g}_n)$ is not nilpotent, contrary to our assumption. From the nilpotence of $Der(\mathfrak{g}_n)$ we deduce $(\mathfrak{g}_n)_0 \neq 0$, so the unitary component is zero. Then every derivation is nilpotent. ■

The obtained definition of characteristic nilpotence is peharps more appropiate for practical purposes. Since any characteristically nilpotent Lie algebra is nilpotent, we search for them on the variety $\mathfrak{N}^n$.

How does characteristic nilpotence behave with direct sums? We will see that characteristic nilpotent Lie algebras can be added to obtain another characteristic nilpotent algebra. Then an $n$-dimensional Lie algebra with this property generates characteristically nilpotent Lie algebras in any dimension $kn$, $k \in \mathbb{Z}^+$. We recall that for a sum $\mathfrak{g} = \bigoplus_{i=1}^{q} \mathfrak{g}_i$ of ideals, the algebra of derivations is given by

$$Der\left(\bigoplus_{i=1}^{q} \mathfrak{g}_i\right) = \bigoplus_{i=1}^{q}\left(Der(\mathfrak{g}_i) \oplus \left(\bigoplus_{i \neq j} \mathcal{D}(\mathfrak{g}_i, \mathfrak{g}_j)\right)\right) \tag{1}$$

where

$$\mathcal{D}(\mathfrak{g}_i, \mathfrak{g}_j) = \{h \in End(\mathfrak{g}) \mid h(\mathfrak{g}_k) = 0 \text{ if } k \neq i,\ h(\mathfrak{g}_i) \subset Z(\mathfrak{g}_i),\ h([\mathfrak{g}_i, \mathfrak{g}_j]) = 0\}$$

**Proposition 14** *Let* $\mathfrak{g} = \bigoplus_{i=1}^{q} \mathfrak{g}_i$ *be a direct sum of ideals. Then* $\mathfrak{g}$ *is characteristically nilpotent if and only if each ideal* $\mathfrak{g}_i$ *is characteristically nilpotent.*

**Proof.** From the characteristic nilpotence of $\mathfrak{g}_i$ it follows that $Z(\mathfrak{g}_i) \subset [\mathfrak{g}_i, \mathfrak{g}_j]$, so that for derivations $f_1 \in Der(\mathfrak{g}_i)$ and $f_2 \in Der(\mathfrak{g}_j)$

$(i \neq j)$ we have $f_1 \circ f_2 = f_2 \circ f_1$ and $h_1 \circ h_2 = 0$ if $h_1 \in \mathcal{D}(\mathfrak{g}_i, \mathfrak{g}_j)$ and $h_2 \in \mathcal{D}(\mathfrak{g}_k, \mathfrak{g}_l)$. Thus every derivation is nilpotent. ■

We are interested in the structural and topological propierties of characteristic nilpotent Lie algebras on the variety $\mathfrak{N}^n$. It is of interest to search their distribution on the different irreducible components of the variety.

**Definition 15** *Let $\mathfrak{g}_n$ be a Lie algebra. A torus of derivations on $\mathfrak{g}_n$ is an abelian subalgebra of $Der(\mathfrak{g}_n)$ formed by semisimple endomorphisms.*

It is known that any two maximal torus of derivations on a Lie algebra $\mathfrak{g}_n$ are conjugate [11], so we need only to deal with conjugation classes. In [19] it is proved that for the nilpotent Lie algebras of dimension $n$ there is only a finite number of conjugation classes of maximal torus $T$. From this we obtain an interesting property for the characteristically nilpotent Lie algebras.

**Proposition 16** *The characteristically nilpotent Lie algebras constitute a constructible set in the variety $\mathfrak{N}^n$ which is empty for $n \leq 6$ and nonempty for any $n \geq 7$.*

**Proof.** Let $T_i$ , $0 \leq i \leq k$ be representatives of conjugation classes of maximal torus on $\mathfrak{N}^n$, where $T_0$ denotes the null torus. Let $\mathfrak{N}^n_{T_i}$ be the closed subset constituted of $T_i$-invariant laws. Then the union of the orbits $\mathcal{O}(\mathfrak{N}^n_{T_i})$ by the action of the linear group $GL(n, \mathbb{C})$ is a constructible set, i.e, a finite union of locally closed subsets. This is complementary to the subset of $\mathfrak{N}^n$ formed by characteristically nilpotent laws, thus they form also a constructible set. ■

**Remark 17** *It is an obvious observation that the characteristically nilpotent Lie algebras don't form a closed subset on $\mathfrak{N}^n$. In fact the Lie algebra $L_n$, which is the model of filiform Lie algebra and defined by the brackets*

$$[X_0, X_i] = X_{i+1}, \ 1 \leq i \leq n-1$$

*in the basis $\{X_0, .., X_n\}$ is not characteristically nilpotent, but we will see that it can be perturbated to obtain characteristically nilpotent Lie algebras.*

## 2.1 Filiform characteristically nilpotent Lie algebras

In this section we give the main results on the construction of parametrized families of characteristically nilpotent filiform Lie algebras. Recall that a law $\mu_n$ is called filiform if $c(\mu_n) = (n-1, 1)$.

Using the Chevalley cohomology of the Lie algebra $\mathfrak{g}_n$ it can be shown that the elements of the space $Z^2(\mathfrak{g}_n, \mathfrak{g}_n)$ correspond to infinitesimal deformations of the algebra $\mathfrak{g}_n = (\mathbb{C}^n, \mu_n)$ ( see [22], [25]). Let $\psi$ be a cocycle and define the operation

$$[x, y]_\psi := [x, y] + \psi(x, y), \quad x, y \in \mathbb{C}^n$$

Then the deformation is linearly expandable if the previous operation satisfies the Jacobi condition, i.e, defines a Lie algebra structure on $\mathbb{C}^n$. It is known that any filiform Lie algebra can be obtained by a linearly expandable deformation of the algebra $L_n$ [40]. Let $\psi \in \bigoplus H_i^2(L_n, L_n) = F_1 H^2(L_n, L_n)$. Then the cocycle admits a decomposition $\psi = \sum_{i=1}^r \psi_i$ with $\psi_i \in H_i^2(L_n, L_n)$. The last nonzero component of this decomposition is called the sill cocycle of $\psi$.

**Lemma 18** *Let $\psi \in F_1 H^2(L_n, L_n)$ be a linearly expandable nonzero cocycle. Then its sill cocycle $\psi_r$ is also linearly expandable.*

**Proof.** From the definition of infinitesimal deformation it follows that $[\,,\,]_\psi$ is linearly expandable if and only if $\forall X, Y, Z \in \mathbb{C}^n$,

$$\psi \circ \psi(X, Y, Z) = \psi(X, \psi(Y, Z)) + \psi(Z, \psi(X, Y)) + \psi(Y, \psi(Z, X)) = 0,$$

As the sill cocycle satisfies this condition, it is linearly expandable. ■

Now let $(L_n)_\psi$ be a deformation with $\psi \in F_1 H^2(L_n, L_n)$. Let $\psi_r$ be the sill cocycle of $\psi$. Then the Lie algebra $(L_n)_{\psi_r}$ is called the sill algebra of $(L_n)_\psi$. The relation between these two algebras is the crucial point to construct characteristically nilpotent Lie algebras[3]

**Theorem 19** *Let $\psi \in F_1 H^2(L_n, L_n)$ be a nonzero linearly expandable cocycle. Then the Lie algebra $(L_n)_\psi$ is characteristically nilpotent if and only if it is not isomorphic to its sill algebra $(L_n)_{\psi_r}$.*

[3]It is evident that the infinitesimal deformations are filiform, for we have seen that the characteristic sequence of the deformation is greater or equal than $c(L_n)$, and this is the maximal one.

A proof of this result can be found in [25].

From the theorem we obtain for example the following characteristic nilpotent Lie algebras with basis $\{X_0, .., X_{2m}\}$ and law

$$\begin{array}{ll} [X_0, X_i] = X_{i+1}, & i = 1, .., 2m-1 \\ [X_1, X_i] = X_{i+3}, & i = 2, .., 2m-3 \\ [X_i, X_{2m-i-1}] = (-1)^{i+1} X_{2m} & i = 1, .., m-1 \end{array}$$

**Remark 20** *For the nonfiliform Lie algebras the determination of characteristically nilpotent Lie algebras is not so well structured. In fact, for any lower characteristic sequence there will appear more naturally graded models than it was the case in the filiform algebras. In addition, the cohomolgy of these models is probably much more complicated than the cohomolgy of $L_n$ or $Q_n$ [39].*

This construction allowed to obtain certain results on the structure of the neighborhhods of filiform Lie algebras on the variety $\mathfrak{N}^n$ [23], so it is of interest for the determination of the irreducible components of the variety of filiform Lie algebra laws, thus for the variety $\mathfrak{N}^n$ itself. We mantain the notation for the cohomology introduced earlier.

**Lemma 21** *Let $s > r, s \neq 2r$. If there is a nonzero cocycle $\psi \in H_s^2(L_n, L_n)$ belonging to $H_s^2(L_n, L_n) \cap B^2\left((L_n)_\psi, (L_n)_\psi\right)$, then this cocycle is unique (up to multiples).*

The proof is based on the structure of the algebra of derivations of a sill algebra and is omitted here. It can be found in [23] and [26]. Now let $A = (L_n)_\psi$ be a filiform algebra, where $\psi \in Z^2(L_n, L_n) \cap F_1 H^2(L_n, L_n)$ and $\psi_r$ denotes the sill cocycle of $\psi$.

**Lemma 22** *Let $n \geq 8$ and $V$ an open set of $\mathfrak{N}^n$ containing $A$. Then there exists a characteristically nilpotent Lie algebra in $V$.*

**Proof.** Let $A$ be a non characteristically nilpotent filiform Lie algebra. From theorem 15 we can suppose $\psi = \psi_r$ with $\psi_r \in H_r^2(L_n, L_n)$. There are four possible cases, namely

1. $r < n-6,\ n-6 \neq 2r$

2. $r < n-6$, $n-5 \neq 2r$
3. $r = n-6$
4. $r > n-6$

We have $\dim H^2_{n-6}(L_n, L_n) = 2$, and any cocycle of this space is also a cocycle of $(L_n)_{\psi_r}$. From the previous lemma there exists a cocycle $\psi_{n-6} \in H^2_{n-6}(L_n, L_n)$, $\psi_{n-6} \notin B^2\left((L_n)_{\psi_r}, (L_n)_{\psi_r}\right)$. Any linear combination of cocycles $\psi_{n-6}$ and $\psi'$, where $\psi'$ is a linearly expandable cocycle belonging to $F_1H^2(L_n, L_n)$ is also linearly expandable. Then a Lie algebra $(L_n)_{\psi(t)}$ with $\psi(t) = \psi_r + t\psi_{n-6}$ $(t \neq 0)$ is not isomorphic to the sill algebra $(L_n)_{\psi_r}$, so it is characteristically nilpotent. The values of $t$ in a neighborhood of 0 give characteristically nilpotent Lie algebras in any neighborhood of $A$. This proves 1. The second assertion is similar. Now let $\psi' \in H^2_{n-5}(L_n, L_n)$ such that its class $[\psi']$ is not zero. We can suppose also that $\psi'$ is a cocycle of $(L_n)_{\psi_r}$ by the previous lemma. Then $(L_n)_{\psi_r + t\psi'}$ is characteristically nilpotent for any nonzero $t$. This proves 3. For the last assertion we can find a cocycle $\psi' \in H^2_{n-5}(L_n, L_n)$ such that $\psi' \notin B^2\left((L_n)_{\psi'}, (L_n)_{\psi'}\right)$ and reasoning as in 1. ■

**Proposition 23** *Let $n \geq 8$ and $C$ an irreducible component of the variety of filiform Lie algebra laws $\mathcal{F}^n$ ( if $n$ is odd we suppose*[1] *$Q_n \notin C$ ). Then $C$ contains a Zariski open subset whose elements are characteristically nilpotent Lie algebras.*

**Proof.** We have seen that the characteristically nilpotent Lie algebras form a constructible set in $\mathfrak{N}^n$. If the assertion is false, then we can find an open set $V$ formed by non characteristically nilpotent Lie algebras, which contradicts the previous lemma. ■

**Corollary 24** *For $\mathfrak{N}^n$ $(n \geq 7)$ there exists an open set whose elements are characteristically nilpotent Lie algebras.* ■

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[1] Recall that the Lie algebra $Q_n$ is defined by

$$[X_1, X_i] = X_{i+1}, \ 2 \leq i \leq n$$
$$[X_i, X_{n+2-i}] = (-1)^i X_{n+1}, \ 2 \leq i \leq n$$

and $n = 2k+1$.

In an analogous way it is possible to prove the stronger version of the previous proposition ( see [23].)

**Proposition 25** *Let $n \geq 8$. Any irreducible component of the variety $\mathcal{F}^n$ contains a nonempty Zariski open subset whose elements are characteristically nilpotent Lie algebras.* ∎

In [13] R.Carles proved the previous corollary for $n = 7$. He also proved that the set $S_7$ of characteristically nilpotent Lie algebras in $\mathfrak{N}^7$ is not open in $\mathfrak{N}^7$. This occurs in fact for any $n > 7$.

Let $\mathfrak{g}_{n,17}$ $(n \geq 8)$ be the Lie algebra defined by the brackets

$$\begin{aligned} [X_1, X_i] &= X_{i+1}, \ 1 \leq i \leq n-1 \\ [X_4, X_2] &= X_n, \\ [X_3, X_2] &= X_{n-1} + X_n \end{aligned}$$

It is immediate that the algebra is filiform and characteristically nilpotent. Let $\psi \in Z^2(\mathfrak{g}_{n,17}, \mathfrak{g}_{n,17})$ be the linear expandable cocycle defined by

$$\begin{aligned} &\psi(X_5, X_3) = X_n, \ \psi(X_5, X_2) = \psi(X_4, X_3) = X_{n-1}, \\ &\psi(X_k, X_2) = 2X_{n-4+\left[\frac{k}{2}\right]}, \ k = 3, 4 \end{aligned}$$

Let $\mathfrak{g}_{n,17} + \varepsilon\psi$ be an infinitesimal deformation of $\mathfrak{g}_{n,17}$.
Now we consider the change of Jordan basis $X_1' = X_1$, $X_2' = X_2 + a_3X_3 + a_4X_4 + a_5X_5$ with the relations

$$\begin{aligned} 1 + a_3^2\varepsilon - 2\varepsilon a_4 &= 0 \\ 3a_5\varepsilon + a_3a_4 - a_3^2\varepsilon - 2\varepsilon a_4 &= 0 \end{aligned}$$

Written in the new basis the algebra $\mathfrak{g}_{n,17} + \varepsilon\psi$ is isomorphic to the Lie algebra $\mathfrak{g}_{n,18}$ defined by

$$\begin{aligned} [X_1, X_i] &= X_{i+1}, \ 1 \leq i \leq n-1 \\ [X_5, X_3] &= \varepsilon X_n \\ [X_5, X_2] &= [X_4, X_3] = \varepsilon X_{n-1} \\ [X_4, X_2] &= 2\varepsilon X_{n-2} \\ [X_3, X_2] &= 2\varepsilon X_{n-3} \end{aligned}$$

From the linear system $(S)$ associated to this algebra we deduce the existence of nonzero eigenvalues for diagonalizable derivations of $\mathfrak{g}_{n,18}$, so it cannot be characteristically nilpotent. We have proven

**Proposition 26** *The set $S_n$ of characteristically nilpotent Lie algebras in $\mathfrak{N}^n$ $(n \geq 8)$ is not open.* ∎

**Remark 27** *The notation we have taken for these algebras is a generalization of the notation used in dimension 8 [4].*

## 2.2 The variety $\mathfrak{N}^8$

The irreducible components of the variety $\mathfrak{N}^8$ were obtained in 1992 with the use of logical calculus [7]. Before that, the nonexistence of a classification for this dimension only allowed to obtain the irreducible components with nonempty intersection with the open set $U^8_{(7,1)}$ of filiform laws [5]. The insufficiency of this result was clear when Seeley [33] gave a component of $\mathfrak{N}^8$ non intersecting the open set $U^8_{(7,1)}$. With the notation used in [7] we give the generating families of the eight irreducible components of $\mathfrak{N}^8$ :

1.

$$\begin{aligned}
\mu^1_\alpha(X_1, X_i) &= X_{i-1},\ 3 \leq i \leq 8 \\
\mu^1_\alpha(X_4, X_8) &= \alpha X_2, \\
\mu^1_\alpha(X_5, X_7) &= X_2, \\
\mu^1_\alpha(X_5, X_8) &= (1+\alpha) X_3 + X_2, \\
\mu^1_\alpha(X_6, X_7) &= X_3, \\
\mu^1_\alpha(X_6, X_8) &= (2+\alpha) X_4 + X_3, \\
\mu^1_\alpha(X_7, X_8) &= (2+\alpha) X_5 + X_4,
\end{aligned}$$

2.

$$
\begin{aligned}
&\mu^2(X_1, X_i) = X_{i-1},\ 3 \le i \le 8 \\
&\mu^2(X_4, X_7) = \mu^2(X_6, X_5) = X_2, \\
&\mu^2(X_4, X_8) = X_3 + X_2, \\
&\mu^2(X_5, X_7) = -\frac{2}{5}X_2, \\
&\mu^2(X_5, X_8) = X_4 + \frac{3}{5}X_3, \\
&\mu^2(X_6, X_7) = -\frac{2}{5}X_3, \\
&\mu^2(X_6, X_8) = X_5 + \frac{1}{5}X_4, \\
&\mu^2(X_7, X_8) = X_6 + \frac{1}{5}X_5
\end{aligned}
$$

3.

$$
\begin{aligned}
&\mu^3_\beta(X_1, X_i) = X_{i-1},\ 4 \le i \le 8 \\
&\mu^3_\beta(X_2, X_6) = X_3, \\
&\mu^3_\beta(X_2, X_7) = X_4 + \beta_1 X_3, \\
&\mu^3_\beta(X_2, X_8) = X_5 + \beta_1 X_4, \\
&\mu^3_\beta(X_5, X_8) = \beta_2 X_3, \\
&\mu^3_\beta(X_6, X_7) = X_3, \\
&\mu^3_\beta(X_6, X_8) = (1+\beta_2) X_4, \\
&\mu^3_\beta(X_7, X_8) = (1+\beta_2) X_5 + \beta_3 X_3
\end{aligned}
$$

4.

$$
\begin{aligned}
&\mu^4_\lambda(X_1, X_i) = X_{i-1},\ 4 \le i \le 8 \\
&\mu^4_\lambda(X_2, X_7) = X_3, \\
&\mu^4_\lambda(X_2, X_8) = X_4 + \lambda_1 X_3, \\
&\mu^4_\lambda(X_5, X_8) = \lambda_2 X_3, \\
&\mu^4_\lambda(X_6, X_7) = \lambda_3 X_3, \\
&\mu^4_\lambda(X_6, X_8) = (\lambda_2 + \lambda_3) X_4 + \lambda_4 X_3, \\
&\mu^4_\lambda(X_7, X_8) = (\lambda_2 + \lambda_3) X_5 + \lambda_4 X_4 + X_2,
\end{aligned}
$$

5.

$$
\begin{aligned}
\mu_\lambda^5(X_1, X_i) &= X_{i-1}, \ 4 \leq i \leq 8 \\
\mu_\lambda^5(X_2, X_8) &= -2X_3, \\
\mu_\lambda^5(X_5, X_7) &= X_3, \\
\mu_\lambda^5(X_5, X_8) &= X_4 + X_2, \\
\mu_\lambda^5(X_6, X_7) &= X_4 + \lambda_1 X_3 - X_2, \\
\mu_\lambda^5(X_6, X_8) &= 2X_5 + \lambda_1 X_4 + \lambda_2 X_3, \\
\mu_\lambda^5(X_7, X_8) &= 2X_6 + \lambda_1 X_5 + \lambda_2 X_4 + \lambda_3 X_3 + \lambda_4 X_2,
\end{aligned}
$$

6.

$$
\begin{aligned}
\mu_\alpha^6(X_1, X_3) &= X_2, \\
\mu_\alpha^6(X_1, X_i) &= X_{i-1}, \ 5 \leq i \leq 8 \\
\mu_\alpha^6(X_2, X_3) &= X_4, \\
\mu_\alpha^6(X_2, X_7) &= \alpha_1 X_4, \\
\mu_\alpha^6(X_2, X_8) &= \alpha_1 X_5 + \alpha_2 X_4, \\
\mu_\alpha^6(X_3, X_6) &= \alpha_3 X_4, \\
\mu_\alpha^6(X_3, X_7) &= (\alpha_1 + \alpha_3) X_5 + \alpha_4 X_4, \\
\mu_\alpha^6(X_3, X_8) &= (2\alpha_1 + \alpha_3) X_6 + (\alpha_2 + \alpha_4) X_5 + \alpha_6 X_4 + \alpha_5 X_2, \\
\mu_\alpha^6(X_6, X_7) &= X_4, \\
\mu_\alpha^6(X_6, X_8) &= X_5 + \alpha_7 X_4, \\
\mu_\alpha^6(X_7, X_8) &= X_6 + \alpha_7 X_5 + \alpha_8 X_4 + (2\alpha_3 + \alpha_1 (2 + \alpha_5)) X_2
\end{aligned}
$$

7.

$$\begin{aligned}
\mu_\beta^7(X_1, X_3) &= X_2, \\
\mu_\beta^7(X_1, X_i) &= X_{i-1},\ 5 \le i \le 8 \\
\mu_\beta^7(X_3, X_7) &= X_4, \\
\mu_\beta^7(X_3, X_8) &= X_5 + \beta_1 X_4 + \beta_2 X_2, \\
\mu_\beta^7(X_5, X_8) &= X_2, \\
\mu_\beta^7(X_6, X_7) &= \beta_3 X_4 - X_2, \\
\mu_\beta^7(X_6, X_8) &= \beta_3 X_5 + X_2, \\
\mu_\beta^7(X_7, X_8) &= \beta_3 X_6 + X_3,
\end{aligned}$$

8.

$$\begin{aligned}
\mu_\beta^8(X_1, X_3) &= X_2, \\
\mu_\beta^8(X_1, X_i) &= X_{i-1},\ 5 \le i \le 8 \\
\mu_\beta^8(X_2, X_8) &= X_4, \\
\mu_\beta^8(X_3, X_7) &= \beta_1 X_4, \\
\mu_\beta^8(X_3, X_8) &= (1 + \beta_1) X_5 + X_4 + \beta_2 X_2, \\
\mu_\beta^8(X_6, X_7) &= \beta_3 X_4, \\
\mu_\beta^8(X_6, X_8) &= \beta_3 X_5 + X_2, \\
\mu_\beta^8(X_7, X_8) &= \beta_3 X_6 + X_3
\end{aligned}$$

with $\alpha_i, \beta_j \in \mathbb{C}$.

**Remark 28** *The two first families are filiform. We observe the importance of the algebra $\mu^2$, for it is the only known rigid law in the variety of nilpotent Lie algebra laws. This algebra is not rigid in $\mathfrak{L}^8$.*
*$\mu^3, \mu^4$ and $\mu^5$ are quasifiliform and the remaining have characteristic sequence* $(5, 2, 1)$.

**Proposition 29** *In the previous notation*

1. *For $\alpha \neq \{-2, -\frac{1}{2}\}$ the laws $\mu_\alpha^1$ are characteristically nilpotent.*

2. *The law* $\mu^2$ *is characteristically nilpotent.*

3. *For* $\beta_2 \neq 0$ *the laws* $\mu^3_\beta$ *are characteristically nilpotent.*

4. *For* $\lambda_2 \neq 0$ *the laws* $\mu^4_\lambda$ *are characteristically nilpotent.*

5. *For* $\lambda_1 \neq 0$ *the laws* $\mu^5_\lambda$ *are characteristically nilpotent.*

6. *For* $\alpha_1 \neq 0, \alpha_2 \neq 0$ *the laws* $\mu^6_\alpha$ *are characteristically nilpotent.*

7. *For* $\beta_3 \neq 0, \beta_1 \neq 1$ *the laws* $\mu^7_\beta$ *are characteristically nilpotent.*

8. *For* $\beta_1 \neq 0$ *the laws* $\mu^8_\beta$ *are characteristically nilpotent.*

**Corollary 30** *Each component of the variety* $\mathfrak{N}^8$ *contains an open set formed by characteristically nilpotent Lie algebras.*

**Remark 31** *We observe the difference between this variety and* $\mathfrak{N}^7$; *in the last there are no dense sets of characteristically nilpotent Lie algebras in the filiform component. For dimensions* $\geq 9$ *the density of the characteristically nilpotent algebras seems to be true at least for the filiform components. Note that this shows that there are sufficient algebras with this property, for the parameters listed in the families are essential and will give pairwise non isomorphic algebras.*

## 2.3 Characteristically nilpotent Lie algebras obtained from nilradicals of Borel subalgebras

This subsection proves that there are sufficiently many characteristically nilpotent Lie algebras. However we observe that these results are of existential nature and therefore non adequate for explicit calculations. Let $\mathfrak{g}$ be a simple Lie algebra of rank $l > 1$ and $\mathfrak{h}$ its Cartan subalgebra, $\Phi$ the root system associated to $\mathfrak{h}$, $\Phi^+$ the system of positive roots relative to a certain ordering and $\Delta$ the system of simple roots. Recall that a Borel subalgebra is a maximal solvable subalgebra of $\mathfrak{g}$.

We consider the subalgebra $B(\Delta) = \mathfrak{h} + \coprod_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ , where $\mathfrak{g}_\alpha$ is the root space corresponding to the root $\alpha$. This subalgebra is a Borel subalgebra of $\mathfrak{g}$ called standard relative to the Cartan subalgebra $\mathfrak{h}$. Now any Borel subalgebra of $\mathfrak{g}$ is conjugated to a standard Borel subalgebra [10], and if $\mathfrak{n}$ denotes the nilradical of an algebra $\mathfrak{g}'$ we have

$\mathfrak{n}(B(\Delta)) = \coprod_{\alpha\in\Phi^+}\mathfrak{g}_\alpha$. Define $\Phi(i)$ as the set of positive roots each of which is the sum of exactly $i$ roots of $\Delta$. Then we can define the $\mathbb{Z}$-graduation on $\mathfrak{n}(B(\Delta D))$ by setting $F_k\mathfrak{n}(B(\Delta)) = \bigoplus_{i\geq k}\mathfrak{n}_i(B(\Delta))$, where $\mathfrak{n}_i(B(\Delta)) = \coprod_{\alpha\in\Phi(i)}\mathfrak{g}_\alpha$. The filtration in the space of cochains is given by

$$F_kC^j(\mathfrak{n},\mathfrak{n}) = \left\{c \in C^j(\mathfrak{n},\mathfrak{n}) \mid c(a_1,..,a_j) \in F_{t_1+..+t_j+k}\mathfrak{n}\right\}$$

whenever $a_i \in F_{t_i}\mathfrak{n}(B(\Delta))$ and where $\mathfrak{n} = \mathfrak{n}(B(\Delta))$.

This filtration extends to the cocycles and coboundaries. Until now we exclude $\mathfrak{g}$ to be a simple algebra of the following types

$$A_i\ \ (1 \leq i \leq 5),\ B_2, B_3, C_3, C_4, D_4, G_2$$

In [27] it is shown that the following system of cocycles suffices for a set of representatives of a basis of $F_0H^2(\mathfrak{n},\mathfrak{n})$ : $\{f_{\alpha,\beta} \mid (\alpha,\beta) \in E\}$ with

$$f_{\alpha,\beta}(x_\gamma, x_\delta) = \begin{cases} x_{\sigma_\alpha\sigma_\beta(\delta)} & \text{for } (\gamma,\delta) = (\alpha, \sigma_\alpha(\beta)) \\ 0 & \text{otherwise} \end{cases}$$

where $\sigma_\alpha$ is the involution associated to the root $\alpha$ and $E$ is the set of pairs of simple roots $(\alpha,\beta)$ in which $(\alpha,\beta)$ is identified with $(\beta,\alpha)$ if $\alpha$ is not joined to $\beta$ in the Dynkin diagram.

**Theorem 32** *Let $\mathfrak{g}$ be a simple Lie algebra and $\mathfrak{n}$ be the nilradical of a Borel subalgebra. Let $\psi = \sum_{\omega\in E}\lambda_\omega f_\omega$ an element of $F_0H^2(\mathfrak{n},\mathfrak{n})$ with $\lambda_\omega \neq 0$ for all $\omega$. Then the Lie algebra $\mathfrak{n}(\psi)$ obtained from the linearly expandable cocycle $\psi$ is characteristically nilpotent.*

The proof of this theorem and the treatment of the above excluded cases can be found in [27]. We only observe that with this information it is possible to construct families of pairwise non isomorphic characteristically nilpotent Lie algebras.

**Remark 33** *This approach to the characteristically nilpotent Lie algebras is most due to Yu. B. Khakimdjanov [27].*

## 2.4 Non-filiform characteristically nilpotent Lie algebras

We have seen in previous sections how to deduce the existence of characteristically nilpotent Lie algebras either from the cohomology of filiform Lie algebras or from nilradicals of Borel subalgebras. However, the second are not constructible explicitely, so we cannot form an image of their structure. In this section we give examples of non-filiform characteristically nilpotent Lie algebras and generalize this concept to the sequence[5] of algebras of derivations. We begin with the $p$-filiform Lie algebras.

**Definition 34** *A nilpotent Lie algebra $\mathfrak{g}_n$ is called p-filiform if its characteristic sequence is $(n-p, 1, \overset{p}{\ldots}, 1)$.*

**Remark 35** *The $(n-1)$-filiform algebras are simply the abelian ones. The $(n-2)$ are direct sums of an Heisenberg algebra $\mathfrak{h}_{2p+1}$ and abelian algebras. The classification of the $(n-3)$-filiform is the first non trivial and can be found in [12]. These algebras arise all as the nilradical of a solvable, rigid law [8]. A classification of the $(n-4)$-filiform can be found in [8], too. The step $(n-5)$ is the first with a nonfinite number of isomorphism classes. It is also the first index for which characteristically nilpotent Lie algebras exist.*

**Proposition 36** *A $(n-5)$-filiform Lie algebra $\mathfrak{g}_n$ is characteristically nilpotent if and only if $\mathfrak{g}_n$ is isomorphic to one of the following laws:*

1. $\mu_1(X_1, X_i) = X_{i+1},\ i \in \{2,3,4,5\};\ \mu_1(X_3, X_2) = X_7;$
   $\mu_1(X_7, X_3) = X_6;\ \mu_1(X_7, X_2) = X_5 + X_6.$

2. $\mu_2^{\alpha}(X_1, X_i) = X_{i+1},\ i \in \{2,3,4,5\};\ \mu_2(X_3, X_2) = X_7 + \alpha X_5\ (\alpha \neq 0);$
   $\mu_2(X_4, X_2) = \alpha X_6;\ \mu_2(X_7, X_3) = X_6;\ \mu_2(X_7, X_2) = X_5 + X_6.$

3. $\mu_3(X_1, X_i) = X_{i+1},\ i \in \{2,3,4,5\};\ \mu_3(X_4, X_2) = X_6;$
   $\mu_3(X_3, X_2) = X_7 + X_5;\ \mu_3(X_7, X_3) = X_6;\ \mu_3(X_7, X_2) = X_5.$

4. $\mu_4(X_1, X_i) = X_{i+1},\ i \in \{2,3,4,5\};\ \mu_4(X_5, X_2) = \mu_4(X_3, X_4) = X_6$
   $\mu_4(X_7, X_2) = X_6;\ \mu_4(X_3, X_2) = X_7.$

[5] Observe that the adjoint representation is not faithful, so the succesive algebras of derivations do not form a tower.

5. $\mu_5(X_1, X_i) = X_{i+1},\ i \in \{2,3,4,5\};\ \mu_5(X_5, X_2) = \mu_5(X_3, X_4) = X_6;$
$\mu_5(X_7, X_3) = X_6;\ \mu_5(X_7, X_2) = X_5 + X_6.$

6. $\mu_6(X_1, X_i) = X_{i+1},\ i \in \{2,3,4,5\};\ \mu_6(X_5, X_2) = \mu_6(X_3, X_4) = X_6$
$\mu_6(X_4, X_2) = \mu_6(X_7, X_3) = X_6;\ \mu_6(X_3, X_2) = \mu_6(X_7, X_2) = X_5.$

7. $\mu_7(X_1, X_i) = X_{i+1},\ i \in \{2,3,4,5\};\ \mu_7(X_3, X_2) = X_7;$
$\mu_7(X_7, X_3) = X_6;\ \mu_7(X_7, X_2) = X_5 + X_6;\ \mu_7(X_8, X_2) = X_6$

8. $\mu_8^\alpha(X_1, X_i) = X_{i+1},\ i \in \{2,3,4,5\};\ \mu_8^\alpha(X_4, X_2) = \alpha X_6;$
$\mu_8^\alpha(X_3, X_2) = X_7 + \alpha X_5\ (\alpha \neq 0);\ \mu_8^\alpha(X_7, X_3) = X_6;$
$\mu_8^\alpha(X_7, X_2) = X_5 + X_6;\ \mu_8^\alpha(X_8, X_2) = X_6.$

9. $\mu_9(X_1, X_i) = X_{i+1},\ i \in \{2,3,4,5\};\ \mu_9(X_4, X_2) = X_6;$
$\mu_9(X_3, X_2) = X_7 + X_5;\ \mu_9(X_7, X_3) = X_6;$
$\mu_9(X_7, X_2) = X_5;\ \mu_9(X_8, X_2) = X_6.$

10. $\mu_{10}(X_1, X_i) = X_{i+1},\ i \in \{2,3,4,5\};\ \mu_{10}(X_5, X_2) = X_6;$
$\mu_{10}(X_3, X_4) = X_6;\ \mu_{10}(X_3, X_2) = X_7;\ \mu_{10}(X_7, X_2) = X_6;$
$\mu_{10}(X_8, X_3) = X_6;\ \mu_{10}(X_8, X_2) = X_5.$

11. $\mu_{11}(X_1, X_i) = X_{i+1},\ i \in \{2,3,4,5\};\ \mu_{11}(X_5, X_2) = \mu_{11}(X_3, X_4) = X_6;$
$\mu_{11}(X_7, X_3) = X_6;\ \mu_{11}(X_7, X_2) = X_5;\ \mu_{11}(X_8, X_2) = X_6;$
$\mu_{11}(X_4, X_2) = X_6;\ \mu_{11}(X_3, X_2) = X_5.$

12. $\mu_{12}(X_1, X_i) = X_{i+1},\ i \in \{2,3,4,5\};\ \mu_{12}(X_5, X_2) = X_6;$
$\mu_{12}(X_3, X_4) = X_6; \mu_{2m}^i(X_7, X_8) = \mu_{12}(X_4, X_2) = X_6;$
$\mu_{12}(X_3, X_2) = X_5;\ \mu_{12}(X_7, X_3) = X_6;\ \mu_{12}(X_7, X_2) = X_5.$

13. $\mu_{13}(X_1, X_i) = X_{i+1},\ i \in \{2,3,4,5\};\ \mu_{13}(X_8, X_3) = X_6;$
$\mu_{13}(X_8, X_2) = X_5;\ \mu_{13}(X_5, X_2) = \mu_{13}(X_3, X_4) = X_6;$
$\mu_{13}(X_3, X_2) = X_7;\ \mu_{13}(X_7, X_2) = X_6;\ \mu_{13}(X_8, X_9) = X_6.$

14. $\mu_{14}(X_1, X_j) = X_{j+1},\ j \in \{2,3,4,5\};\ \mu_{14}(X_5, X_2) = \mu_{14}(X_3, X_4) = X_6;$
$\mu_{14}(X_7, X_j) = X_{j+3},\ j \in \{2,3\};\ \mu_{14}(X_8, X_2) = \mu_{14}(X_7, X_9) = X_6;$
$\mu_{14}(X_4, X_2) = X_6;\ \mu_{14}(X_3, X_2) = X_5.$

The proof follows immediately from the classification of the $(n-5)$-filiform Lie algebras [9].

**Corollary 37** *There are characteristically nilpotent Lie algebras* $\mathfrak{g}_n$ *of nilindex* 5 *for* $n = 7, 8, 9, 15, 16, 17, 18$ *and* $n \geq 21$.

**Remark 38** *We observe that in particular a nonsplit* $(n-5)$*-filiform characteristically nilpotent Lie algebra is* 2*-abelian, i.e, the ideal* $C^2\mathfrak{g}_n$ *of the descending central sequence is abelian while the derived algebra is not abelian. Starting with this property, we are able to construct characteristically nilpotent Lie algebras via deformation theory:*

**Definition 39** *Let* $\mathfrak{g}_n$ *be an* $n$*-dimensional nilpotent Lie algebra. We call commutativity index of* $\mathfrak{g}_n$ *to the smallest positive integer* $k$ *such that the ideal* $C^k\mathfrak{g}_n$ *of the descending central sequence is abelian.*

**Definition 40** *A nilpotent Lie algebra* $\mathfrak{g}_n$ *is called* $k$*-abelian if its commutativity index is exactly* $k$.

**Remark 41** *In [21] we can found a weaker definition of* $k$*-abelianity. In fact, for their definition any* $k$*-abelian Lie algebra is contained in the set of* $(k+1)$*-abelian Lie algebras. This is the fundamental difference between that and our definition: for us the* $k$*-abelian algebras are not contained in the* $(k+1)$*-abelian algebras, for the last are not allowed to have the* $k^{th}$ *ideal abelian. Thus we are "separating" the algebras accordind to their commutativity propierties in the central descending sequence.*

**Proposition 42** *Let* $\mathfrak{g}_{2m+2}$ $(m \geq 4)$ *be the Lie algebra defined by the brackets*

$$\begin{aligned}
[X_1, X_i] &= X_{i+1},\ 2 \leq i \leq 2m-1 \\
[X_1, X_{2m+1}] &= X_{2m+2} \\
[X_2, X_3] &= X_{2m+1}, \\
[X_2, X_4] &= X_{2m+2} \\
[X_2, X_{2m+1}] &= X_{2m-1} \\
[X_2, X_{2m+2}] &= X_{2m}, \\
[X_j, X_{2m+1-j}] &= (-1)^j X_{2m},\ 2 \leq j \leq \left[\frac{2m+1}{2}\right]
\end{aligned}$$

*Then* $\mathfrak{g}_{2m+2}$ *is characteristically nilpotent with characteristic sequence* $(2m-1, 2, 1)$.

**Proof.** Let $f(X_i) = \sum_{i=1}^{2m+2} f_i^j X_j$ be a derivation of $\mathfrak{g}_{2m+2}$. As the vectors $X_1, X_2$ generate the Lie algebra, it is immediate that the coefficient $f_i^i$ of $f(X_i)$ for $i \geq 3$ is a linear combination of the coefficients $f_1^1, f_2^2$ and $f_1^2$. Thus, if the algebra is characteristically nilpotent, it is enough to prove that the derivation induced induced in the quotient Lie algebra $\frac{\mathfrak{g}_{2m+2}}{C^1\mathfrak{g}_{2m+2}}$ is zero. Now the bracket $[X_j, X_{2m+1-j}] = (-1)^j X_{2m}$ implies $f_2^2 = f_1^1$ and the bracket $[X_2, X_3]$ implies $f_1^2 = 0$. Using the remaining bracket $[X_2, X_{2m-1}]$ we obtain $f_1^1 = 0$, so that the restriction is zero and the derivation has no nonzero diagonal entries. ∎

**Remark 43** *If we consider the ideal* $\langle X_{2m+2}\rangle$, *the quotient algebra* $\frac{\mathfrak{g}_{2m+2}}{\langle X_{2m+2}\rangle}$ *is an* $(m-1)$*-abelian Lie algebra with characteristic sequence* $(2m-1,1,1)$. *It is not difficult to see that this algebra is also characteristically nilpotent.*

## 2.5 Characteristically nilpotent Lie algebras of index 2

The determination of the structure of the algebra of derivations of a nilpotent Lie algebra is a very difficult problem, and no criterions exist to characterize algebras with specific structure on its algebra of derivations. From the existence of characteristically nilpotent Lie algebras it follows immediately the following question: how far is it possible to generalize this concept? S. Tôgô asked in 1961 for the existence of characteristically nilpotent Lie algebras of derivations. In this section we give a positive answer to this question.

**Lemma 44** *If* $\mathfrak{g}$ *is a Lie algebra such that* $Der(\mathfrak{g})$ *is characteristically nilpotent, then* $\mathfrak{g}$ *is characteristically nilpotent.*

**Proof.** If $Der(\mathfrak{g})$ is characteristically nilpotent, then it is nilpotent. In particular the image $ad\,\mathfrak{g}$ of the adjoint representation is nilpotent, so $\mathfrak{g}$ is nilpotent. It is evident that $\mathfrak{g}$ is characteristically nilpotent. ∎

**Definition 45** *A Lie algebra* $\mathfrak{g}$ *is called characteristically nilpotent of index* 2 *if* $Der(\mathfrak{g})$ *is a characteristically nilpotent Lie algebra.*

**Remark 46** *As characteristically nilpotent Lie algebra over* $\mathbb{C}$ *do not exist for dimensions* $\leq 6$, *we conclude that algebras with index* 2 *must have at least dimension* 9.

**Example 47** *Let $\mathfrak{g}$ the Lie algebra with associated law $\mu_5$ defined in the previous section. The algebra of derivations $Der(\mathfrak{g})$ has dimension* 10 *and is isomorphic to*

$$\begin{array}{lll} [Z_1,Z_2]=Z_3, & [Z_2,Z_6]=-Z_5, & [Z_7,Z_8]=2Z_5-2Z_6+2Z_{10} \\ [Z_1,Z_3]=Z_4, & [Z_2,Z_8]=-Z_6, & [Z_7,Z_9]=Z_5-2Z_6+2Z_{10} \\ [Z_1,Z_4]=Z_5, & [Z_2,Z_9]=-Z_4-2Z_6, & [Z_8,Z_9]=2Z_6-2Z_{10} \\ [Z_1,Z_7]=-Z_4, & [Z_2,Z_{10}]=-Z_5, & \\ [Z_1,Z_8]=-Z_6, & [Z_3,Z_8]=-Z_5, & \\ & [Z_3,Z_9]=-Z_5, & \end{array}$$

*This algebra is easily seen to be characteristically nilpotent.*

This and other found algebras with this property have a common property: there is always an outer derivation $\theta$ which belongs to the derived algebra of the algebra of derivations. However, this condition is not sufficient. For example, the Lie algebra $(\mathbb{C}^7,\mu_6)$ defined in the previous section has not a characteristically nilpotent Lie algebra of derivations, and there are outer derivations in $C^1(Der(\mu_6))$. On the other side, it is obvious that if an outer derivation $\theta$ belongs to the center of the algebra of derivations, the last is not characteristically nilpotent.

**Conjecture 48** *If $\mathfrak{g}$ has a characteristically nilpotent Lie algebra of derivations, then there exist outer derivations $\theta_1,\theta_2,\theta_3$ such that*

$$[\theta_1,\theta_2]=\lambda\theta_3 \pmod{IDer(\mathfrak{g})}$$

*where $\lambda\in\mathbb{C}-\{0\}$ and $IDer(\mathfrak{g})$ denotes the inner derivations.*

**Remark 49** *In* 1970 *Dyer gave the first example of a characteristically nilpotent Lie algebra with unipotent automorphism group in dimension nine. Two years later, Favre [18] constructed an example in dimension 7. The previous example behaves as the original of Dixmier and Lister, and has not an unipotent automorphism group. We note that the algebra of Lister and Dixmier has not a characteristically nilpotent Lie algebra of derivations.*

Having recovered the problem traced by Tôgô in his article [37], very little is known about characteristically nilpotent derivation algebras. We

can generalize the concept by saying that a Lie algebra is characteristically nilpotent of index $k$ if the $(k-1)^{th}$ algebra of derivations is characteristically nilpotent. Then we have a sequence of derivation algebras. Now, how long can this sequence be? Are there nilpotent Lie algebras with infinite sequences of such type? The density of the characteristically nilpotent Lie algebras in irreducible components suggests a positive answer, but we haven't found any construction yet. Moreover, there seems to be no relation between the characteristic sequence of a characteristically nilpotent algebra $\mathfrak{g}$ and the characteristic sequence of its algebra of derivations: it is not difficult to construct an example which preserves the nilindex.
Finally, it would be interesting to know if there exist irreducible components of the variety $\mathfrak{N}^n$ which are the Zariski closure of a family of characteristically nilpotent Lie algebras of derivations.

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